The Poisson Equation with Local Nonregular Similarities

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Received 18 June 2000; accepted 30 October 2000

Moffatt and Duffy [1] have shown that the solution to the Poisson equation, defined on rectangular domains, includes a local similarity term of the form: $r^2 log(r) cos(2\theta)$. The latter means that the second (and higher) derivative of the solution with respect to r is singular at $r = 0$. Standard high-order numerical schemes require the existence of high-order derivatives of the solution. Thus, for the case considered by Moffatt and Duffy, the high-order finite-difference schemes loose their high-order convergence due to the nonregularity at $r = 0$. In this article, a simple method is outlined to regain the high-order accuracy of these schemes, without the need of any modification in the scheme's algorithm. This is a significant consideration when one wants to use a given finite-difference computer code for problems with local nonregular similarity solutions. Numerical examples using the modified scheme in conjunction with a sixth-order finite difference approximation are provided. © 2001 John Wiley & Sons, Inc. Numer Methods Partial Differential Eq 17: 336–346, 2001

Keywords: high-order scheme; Poisson's equation; local nonregular similarity

I. INTRODUCTION

Many numerical schemes, aimed at solving partial differential equations describing conservation laws of fluid dynamics, heat transfer, and solid mechanics, involve the solution of the Poisson equation. Finite-difference approximations to the Poisson equation lead to large sparse systems of linear equations, requiring time-consuming algorithms for their solution. One of the remedies to this situation is the use of high-order schemes that provide higher accuracy on coarse meshes, and employ fast methods for the solution of the system of equations. In [2], for example, fast and stable direct methods have been developed for solving Poisson's difference equations over *a rectangular domain*. Recently, a high-order scheme for the solution of the Poisson equation over a rectangle (2-dimensional domain) has been presented in [3]. This scheme, and many other high-order standard schemes, loose their high-order accuracy for a wide variety of Poisson problems

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that contain singular points, i.e., distinct points at which one of the derivatives of the exact solution is infinity. Herein, a simple method is outlined to regain the high-order accuracy of these schemes, without the need of any modification in the scheme's algorithm. This is a significant consideration when one wants to use a given finite-difference computer code for problems with local similarity solutions.

Consider the Poisson equation over a 2-dimensional domain Ω:

$$\nabla^2 w = -\beta_0 \quad \text{in } \Omega, \tag{1}$$

where β_0 is a given constant. Homogeneous Dirichlet boundary conditions are imposed on the boundary $\partial\Omega$:

$$w = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Moffatt and Duffy [1] have shown that, if the boundary has a sharp corner of angle 2α (see Fig. 1), then the solution is $w^{(0)}(r,\theta)$, depending only on r, θ and α (i.e., independent of the "remote" geometry):

$$w^{(0)}(r,\theta) = -\frac{\beta_0 r^2}{4}\left(1 - \frac{cos 2\theta}{cos 2\alpha}\right), \tag{3}$$

where a polar coordinate system (r,θ) is located in the corner with $\theta = 0$ along the bisector. For the specific angle $2\alpha = \frac{\pi}{2}$, the similarity solution (3) "explodes," which requires special treatment.

To resolve this case, Moffatt and Duffy [1] considered the general solution to (1) satisfying $w = 0$ on $\theta = \pm\alpha$:

$$w = w^{(0)} + \sum_{n=0}^{\infty} A_n r^{\lambda_n} cos \lambda_n \theta, \tag{4}$$

where

$$\lambda_n = (2n+1)\frac{\pi}{2\alpha}, \quad n = 0, 1, 2, ... \tag{5}$$

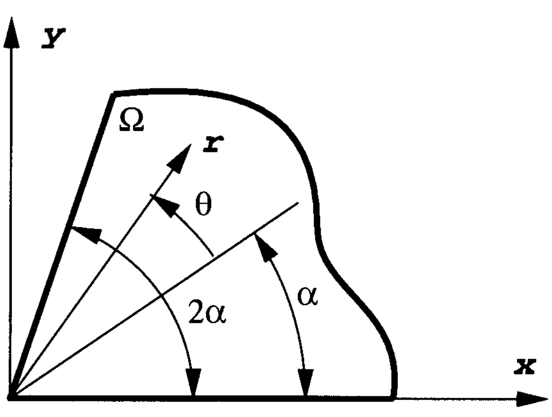

FIG. 1. Notations.

Generally speaking, the coefficients A_n depend on the domain geometry and boundary condition away from the origin. For the *ad hoc* problem, when the domain is bounded by the lines $\theta = \pm\alpha$ and the circular arc $r = a$, the coefficients A_n are determined by the condition $w = 0$ on $r = a$, and, in particular,

$$A_0 = \frac{2\beta_0 a^{2-\lambda_0}}{\alpha\lambda_0(\lambda_0^2 - 4)}. \tag{6}$$

Rewriting (4) for the special case with $2\alpha = \frac{\pi}{2} + \epsilon$ and considering the limiting case $\epsilon \to 0$ yields [1]

$$w = w_s + \sum_{n=1}^{\infty} A_n r^{\lambda_n} cos\lambda_n\theta, \tag{7}$$

where

$$w_s = -\frac{\beta_0 r^2}{\pi}\left[\frac{\pi}{4} + log(r)cos2\theta - \theta\sin 2\theta\right]. \tag{8}$$

(The derivation in [1] leads to the expression with $log(\frac{r}{a})$. The term with $log(a)$, which is a particular solution of Laplace's equation, is omitted in (8) to detach a purely local similarity solution independent of the global geometry.)

The solution (8) consists of the particular solution to (1) $(-\beta_0 r^2/4)$ and the local similarity solution, which is a solution to Laplace's equation. The leading term of the local similarity solution near $r = 0$ is of the form: $r^2 log(r)cos2\theta$. The latter means that the second (and higher) derivative of w with respect to r is singular at $r = 0$, though this singularity is compensated by other terms of the Laplacian operator to provide the solution to (1). We call this singularity a *hidden singularity*, because it is not anticipated in general: the solution and its first-order derivative are finite, but its higher-order derivatives become infinite at the vicinity of a singular point at the boundary.

The term *regular* (or *holomorphic*) is used for functions that are expandable in the Taylor series, i.e., for functions that are differentiable with *all* their derivatives. To denote the singularity of high-order derivatives of the similarity solution (8), we use the term *nonregular*. We recall that the leading term of the similarity solution depends on the right-hand side of the Poisson equation, and the expression (8) has been *ad hoc* obtained for (1). In the Appendix, we consider a special case with another type of nonregular similarity solution.

High-order numerical schemes are based on the Taylor series expansion of the solution, and, therefore, require the existence of high-order derivatives of the sought solution. Thus, for the case $2\alpha = \pi/2$ considered above, standard high-order finite-difference schemes, as the one developed in [3], loose their high-order convergence due to the *hidden singularity* at $r = 0$. The singularity of the high-order derivatives contaminates the global accuracy of the numerical solution. For example, a high-order scheme, which requires existence of the fourth-order derivative of the solution, would fail for the problem at hand. For instance, in [3] this case has been considered and the numerical solution, obtained with the sixth-order scheme, did not show the sixth-order accuracy.

Herein, a simple method is outlined to regain the high-order accuracy of the scheme developed in [3], and similar schemes, for solving the Poisson equation without the need of any modification in the scheme's algorithm.

II. NUMERICAL METHOD

Consider the Poisson equation on a square domain: $\Omega = \{(x, y)|0 \leq x \leq 2,\ \ 0 \leq y \leq 2\}$:

$$\nabla^2 w = f(x, y), \tag{9}$$

where $f(0, 0) = -\beta_0 \neq 0$. The later means that the sought solution to (9) possesses the nonregular similarity property at the corner $(0, 0)$.

For the sake of simplicity, we assume that $f(x, y)$ is symmetric about $x = 1$ and $y = 1$. Thus, the boundary conditions are

$$\begin{aligned} &w(0, y) = w(x, 0) = 0 \\ &\left.\frac{\partial w}{\partial x}\right|_{y=1} = 0, \\ &\left.\frac{\partial w}{\partial y}\right|_{x=1} = 0. \end{aligned} \tag{10}$$

The solution $w(x, y)$ can be decomposed into singular and regular parts:

$$w(x, y) = w_s(x, y) + w_R(x, y), \tag{11}$$

where the similarity nonregular solution w_s is given in (8) and w_R is an unknown *regular* function. Substituting (11) into (9), we obtain the Poisson equation for w_R:

$$\nabla^2 w_R = f(x, y) + \beta_0, \tag{12}$$

subject to the Dirichlet (at walls)

$$w_R(0, y) = w_R(x, 0) = 0, \tag{13}$$

and Neumann (symmetry)

$$\left.\frac{\partial w_R}{\partial x}\right|_{x=1} = -\left.\frac{\partial w_s}{\partial x}\right|_{x=1} \overset{\text{def}}{=} p(y), \qquad \left.\frac{\partial w_R}{\partial y}\right|_{y=1} = -\left.\frac{\partial w_s}{\partial y}\right|_{y=1} \overset{\text{def}}{=} q(x) \tag{14}$$

boundary conditions. Using (8) for the definitions of $p(y)$ and $q(x)$, we find:

$$\begin{aligned} p(x) &= \frac{1}{\pi}[2tan^{-1}(x) + xln(1 + x^2) + x], \\ q(y) &= \frac{1}{\pi}[2tan^{-1}(y) + yln(1 + y^2) + y]. \end{aligned} \tag{15}$$

Finally, solving (12) subject to the boundary conditions in (13)-(14) for the regular function w_R by high-order schemes ensures high-order accuracy convergence. Once w_R is found, the overall solution w is constructed by (11).

In this study, we apply the high-order discretization scheme described in [3] for the Poisson equation defined on a rectangular domain. Let a square $\Omega = \{0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ be a 2-dimensional computational domain. The grid points on the domain are defined by two discrete equally spaced point sets:

$$\omega_x \overset{\text{def}}{=} \{x(i) = ih_x,\ i = 1,\ 2, ...,\ N_x;\ (N_x - 1)h_x = 1\}, \tag{16}$$

$$\omega_y \stackrel{\text{def}}{=} \{y(j) = jh_y,\ j = 1,\ 2, \ldots,\ N_y;\ (N_y - 1)h_y = 1\}. \tag{17}$$

The nodes $(1, j)$, $(i, 1)$ and (N_x, j), (i, N_y) are boundary nodes, where the Dirichlet and Neumann boundary conditions are imposed, respectively (see Fig. 2). The finite-difference equation for (12) on a 9-point discretization stencil is of the form

$$a(w_R)_{i,j} + bS^{(xy)}_{i,j} + cS^{(x)}_{i,j} + dS^{(y)}_{i,j} = r_{f+\beta_0}, \tag{18}$$

where

$$S^{(xy)}_{i,j} \stackrel{\text{def}}{=} (w_R)_{i-1,j-1} + (w_R)_{i-1,j+1} + (w_R)_{i+1,j-1} + (w_R)_{i+1,j+1}$$

$$S^{(x)}_{i,j} \stackrel{\text{def}}{=} (w_R)_{i-1,j} + (w_R)_{i+1,j}, \qquad S^{(y)}_{i,j} \stackrel{\text{def}}{=} (w_R)_{i,j-1} + (w_R)_{i,j+1}$$

$$r_\psi \stackrel{\text{def}}{=} \psi_{ij} + \frac{1}{4!}(h_x^2 + h_y^2)(D_x^2 + D_y^2)\psi_{ij} + \frac{2}{6!}(h_x^4 D_x^4 + 4h_x^2 h_y^2 D_x^2 D_y^2 + h_y^4 D_y^4)\psi_{ij}$$

$$+\frac{1}{4!}(h_x^2 - h_y^2)(D_x^2 - D_y^2)\psi_{ij} + \frac{3}{2\cdot 6!}(h_x^2 - h_y^2)^2 D_x^2 D_y^2 \psi_{ij}.$$

The scheme coefficients are

$$a = -2\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right) + 4b, \quad b = \frac{2}{4!}\left(\frac{1}{h_x^2} + \frac{1}{h_y^2}\right),$$

$$c = \frac{1}{h_x^2} - 2b\ ,\ d = \frac{1}{h_y^2} - 2b.$$

To apply the discrete Poisson equation (18) on the symmetry lines $i = N_x$ and $j = N_y$, one needs the values of w_R at the fictitious (dummy) points of the computational domain: $(w_R)_{N_x+1,j}$ and $(w_R)_{i,N_y+1}$. Using the sixth-order approximation to the Neumann boundary condition derived in [4], we have:

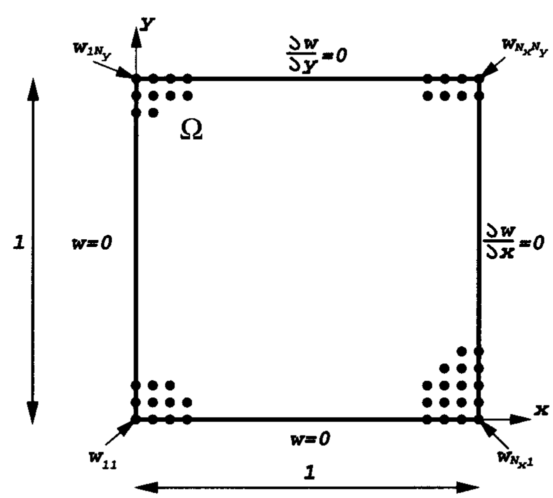

FIG. 2. Computational domain and boundary conditions.

$$(w_R)_{N_x+1,j} = (w_R)_{N_x-1,j} + 2h_x p(y) - \frac{2}{3!}h_x^3 \frac{d^2 p}{dy^2} + \frac{2}{5!}h_x^5 \frac{d^4 p}{dy^4}, \tag{19}$$

$$(w_R)_{i,N_y+1} = (w_R)_{i,N_y-1} + 2h_y q(x) - \frac{2}{3!}h_y^3 \frac{d^2 q}{dx^2} + \frac{2}{5!}h_y^5 \frac{d^4 q}{dx^4}. \tag{20}$$

The equations (18)-(20) together with (15) complete the formulation of the high-order discretization scheme for the Poisson equation (12). The solution to the original Poisson equation is defined by the known similarity nonregular solution w_s, added to the regular solution w_R. The truncation error for the suggested scheme is of the sixth-order $O(h^6)$ on a square mesh ($h_x = h_y = h$) and of the fourth-order $O(h_x^4,\ h_x^2 h_y^2,\ h_y^4)$ on an unequally spaced mesh [3].

III. NUMERICAL EXAMPLES

In this section, we present numerical results obtained using the suggested high-order method, compared to the standard high-order method. The results are compared with exact solutions when available. Using a square mesh ($h_x = h_y = h \;\; N_x = N_y = N$), the suggested method is expected to provide the sixth-order $O(h^6)$ accuracy. In order to conduct accurate comparison and because the expected errors are very small (less than $\mathcal{O}(10^{-12})$), we use double-precision accuracy in our computations. The system of algebraic equations is solved by the Gauss–Seidel iteration method with a convergence tolerance $\epsilon = 10^{-14}$. Two examples are considered, the first with a constant right-hand side in the Poisson equation, and the another with a smooth function on the right-hand side being constant at the origin, thus exciting the ''hidden singularity'' in the solution.

A. Example 1: $f(x, y) = -\beta_0 = \text{const}$, duct flow

This example describes Poiseuille flow of an incompressible fluid, forced under a pressure difference to move in a rectangular duct of constant cross-sectional shape: $0 \le x \le 2a,\ 0 \le y \le 2b$. The boundary conditions are given in (10), only that $y = 1$ should be replaced by $y = b$, and $x = 1$ should be replaced by $x = a$. The exact solution for this well-known problem is readily derived by the method of separation of variables and in the literature there are different forms of this solution. We use the solution in the form

$$w(x, y) = \sum_{m=1}^{\infty} U_m(x) \sin(\lambda_m y), \tag{21}$$

$$U_m(x) = \frac{4\beta_0}{\pi(2m-1)\lambda_m^2} \left[1 - \frac{\cosh \lambda_m (x-a)}{\cosh \lambda_m a}\right], \;\; \lambda_m = \frac{\pi(2m-1)}{2b}. \tag{22}$$

In the practical computation of the exact reference solution, we chose the number of terms in the series to make the remainder smaller than 10^{-14}. To illustrate the rate of convergence of the proposed method compared to the standard method, we plot in Fig. 3 the error of the finite difference solution vs. the grid spacing (h). We define the discrete RMS norm as follows:

$$\|e\|_{rms} = \|w - w^{(h)}\|_{rms} \stackrel{\text{def}}{=} \sqrt{\frac{1}{(N_x-1)(N_y-1)} \sum_{i,j=1}^{N_x,N_y} \left[w_{ij} - w_{ij}^{(h)}\right]^2}, \tag{23}$$

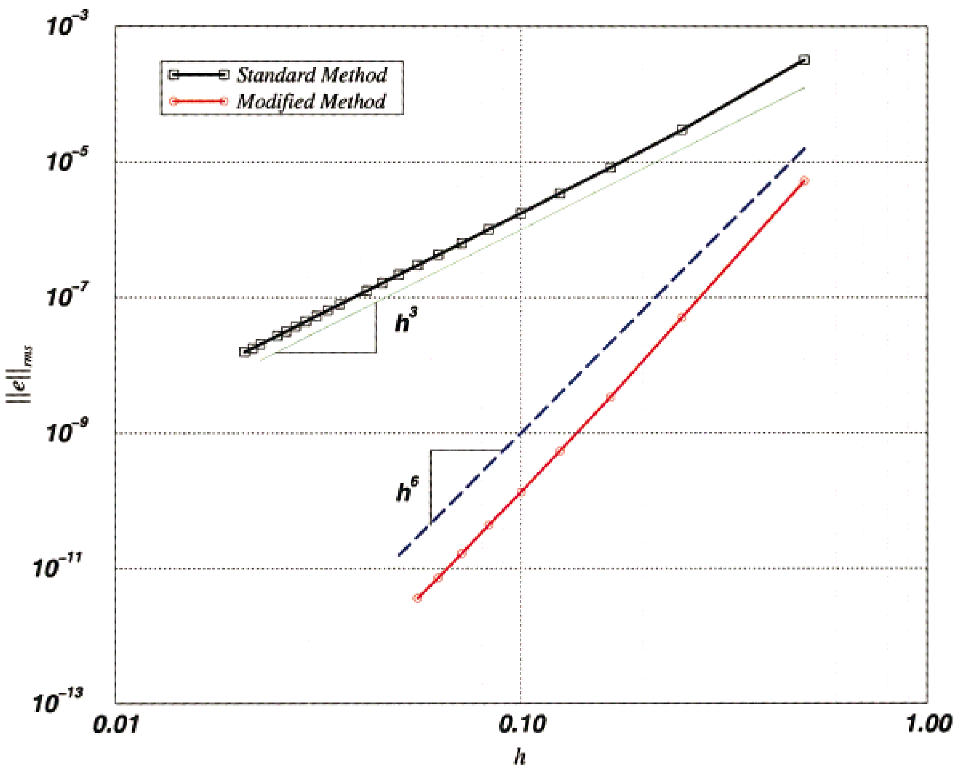

FIG. 3. Convergence rates of the standard and modified finite-difference schemes for Example 1. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

where $w_{ij}^{(h)}$ is the finite difference solution at grid points i, j for the grid spacing h. The exact solution w_{ij} is evaluated from (22) at each point (x_i, y_j) coinciding with the i, jth node. We consider a square ($a = b = 1$) duct, and equally spaced grid points, $h_x = h_y = h = 1/N$. The high-order scheme developed in [3] generates an accuracy of approximately $\mathcal{O}(h^3)$ and not of $\mathcal{O}(h^6)$ because of the nonregular similarity at the origin, as predicted. On the other hand, Fig. 3 clearly illustrates that the modified method regains the sixth-order convergence rate: the slope of the error curve is equal to 6.

B. Example 2: $\mathbf{f(x, y) = \cos(\omega_{xy}),\ \ \omega_{xy} = \frac{\pi}{2} x(2 - x)y(2 - y),\ \ 0 \leq x \leq 1, 0 \leq y \leq 1}$

Herein we consider a problem with a smooth function on the right-hand side of the Poisson equation, which is equal to a constant at the origin $(0, 0)$. The latter excites the nonregular similarity in the exact solution. The boundary conditions are as given in (10). Because an exact solution for this example is not available, we obtain a numerical reference solution, which, for all practical purposes, can be considered as *the exact solution* for the problem at hand. The high and sufficient accuracy of the reference solution is obtained as follows. For different grid spacings h, we compute numerical solutions, using both the standard and modified high-order methods. We define the following integral:

$$I^{(h)} = \sqrt{\iint_{\Omega} \left(w^{(h)}\right)^2 dxdy}. \tag{24}$$

Due to consistency of the considered problem, the value of $I^{(h)}$ converges to a constant as the mesh size $h \to 0$. In Table I, we summarize the values of $I^{(h)}$. The numerical integration used the seventh-order Newton–Cotes formula [5] to capture the sixth-order accuracy of the modified scheme. The results obtained with $N = 33$ for the modified scheme (i.e., $h = 0.03125$ and the expected error is $h^6 \approx 10^{-9}$) are considered by us as the reference (''exact'') for comparison.

Table I shows that the numerical solution obtained by the modified method with $N = 33$ or the one obtained with the standard method with $N = 513$, where the integral is identical up to the eighth digit could be considered as the reference solution. We computed the RMS norm of the errors obtained using the modified as well as the standard high-order scheme in [3]. These are shown in Fig. 4. Again, it is clearly seen that the standard high-order method does not perform satisfactorily, while the modified high-order scheme regains the sixth-order convergence rate: the slope of the error curve is equal to 6.

IV. APPENDIX: HIGH-ORDER LOCAL NONREGULAR SIMILARITIES

Consider the Poisson equation over a 2-dimensional domain Ω (see Fig. 1):

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = f(r,\theta) \quad \text{in } \Omega \tag{A1}$$

subject to homogeneous Dirichlet boundary conditions on the boundary $\partial\Omega$. We assume that the boundary has a sharp corner of angle 2α. Thus, (r, θ) are polar coordinates with the origin at the corner as shown in Fig. 1.

Imposing the boundary condition $w = 0$ at $\theta = \pm\alpha$, leads to the following set of eigenvalues and eigenfunctions:

$$\lambda_n = (2n+1)\frac{\pi}{2\alpha}, \quad \{cos\lambda_n\theta\}, \quad n = 0, 1, 2, ... \tag{A2}$$

Let us assume that the right-hand of (A1) may be represented as a series of the form

TABLE I. Values of $I^{(h)}$ computed using the modified and standard methods.

N	h	$I^{(h)}$-modified meth.	$I^{(h)}$-standard meth.
5	0.25000	0.0064403219	0.006103699
9	0.12500	0.0064028003	0.006319438
17	0.06250	0.0064024646	0.006381731
33	0.03125	0.0064024584	0.006397282
65	0.01563		0.006401164
129	0.00781		0.006402134
257	0.00391		0.006402377
513	0.00195		0.006402438

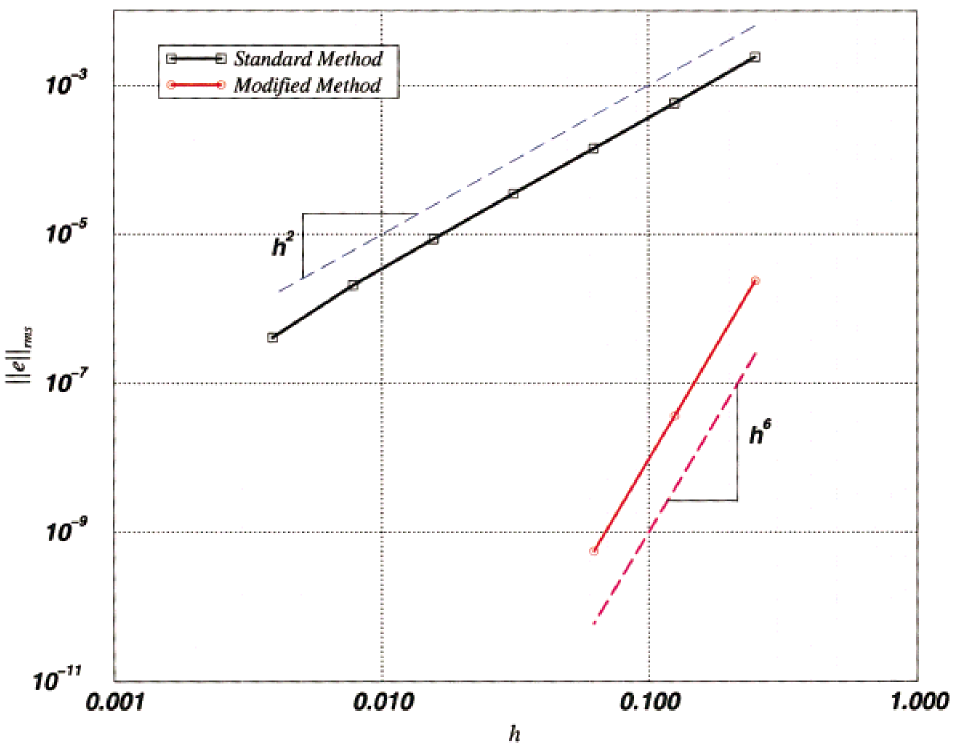

FIG. 4. Convergence rates of the standard and modified finite-difference schemes for Example 2. [Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com.]

$$f(r,\theta) = \sum_{n=0}^{\infty} C_n r^{\lambda_n - 2} cos(\lambda_n - 2)\theta. \tag{A3}$$

The solution of (A1) then is

$$w(r,\theta) = \sum_{n=0}^{\infty} w^{(n)} + \sum_{n=0}^{\infty} A_n r^{\lambda_n} cos\lambda_n\theta, \tag{A4}$$

where $w^{(n)}$ is *the particular* solution to (A1) *corresponding* to the *n*-th term of the series (A3). The second series in (A4) is the superposition of Laplace's equation solutions, where the coefficients A_n depend on the domain's global geometry.

It will be shown that considering only the first two terms in (A3) is sufficient, namely:

$$f(r,\theta) = C_0 + C_1 r^4 cos4\theta. \tag{A5}$$

It is readily seen that the corresponding particular solutions are

$$w^{(0)}(r,\theta) = C_0 \frac{r^2}{4}\left(1 - \frac{cos2\theta}{cos2\alpha}\right), \quad w^{(1)}(r,\theta) = C_1 \frac{r^6}{20}\left(cos4\theta + \frac{cos6\theta}{cos6\alpha}\right), \tag{A6}$$

where $2\alpha = \frac{\pi}{2} - \epsilon$. These solutions are nonregular for $\theta = \pm\alpha = \pm\frac{\pi}{4}$. The solution $w^{(0)}$ has been addressed in the article and analyzed in detail in [1]. Herein we consider $w^{(1)}$.

For the sake of developing the limiting case $\alpha \to \frac{\pi}{4}$, we assume that the domain is bounded by the lines $\theta = \pm\frac{\pi}{4}$ and the circular arc $r = a$. The coefficients A_n are determined by the condition $w = 0$ on $r = a$. Using the ortogonality of the eigenfunctions (A2), we obtain

$$A_n = (-1)^{n+1} \frac{2C_1 a^{6-\lambda_n}\lambda_n}{\alpha(\lambda_n^2 - 16)(\lambda_n^2 - 36)}. \qquad (A7)$$

Expressing λ_1 from (A1) as $\lambda_1 = 6 + \frac{12\epsilon}{\pi}$ and adding the second term ($n = 1$) of the series in (A4) to the solution $w^{(1)}$, we find

$$w(r,\theta) = w_s^{(1)} + \sum_{n=2}^{\infty} A_n r^{\lambda_n} cos\lambda_n\theta, \qquad (A8)$$

where

$$w_s^{(1)}(r,\theta) = C_1 \frac{r^6}{20}\left\{cos4\theta - \frac{1}{3\epsilon}\left[cos6\theta - \left(\frac{r}{a}\right)^{\frac{12\epsilon}{\pi}} cos\left(6 + \frac{12\epsilon}{\pi}\right)\theta\right]\right\}. \qquad (A9)$$

Finally, at the limit $\epsilon \to 0$ (i.e., $2\alpha = \frac{\pi}{2}$), the local similarity solution reads

$$w_s^{(1)}(r,\theta) = C_1 \frac{r^6}{5\pi}\left[\frac{\pi}{4}cos4\theta + log(r)cos6\theta - \theta\sin 6\theta\right]. \qquad (A10)$$

(The derivation of (A10) from (A9) leads to the expression with $log(\frac{r}{a})$. The term with $log(a)$, which is a particular solution of Laplace's equation, is omitted in (A10) to detach the purely local similarity solution independent of the global geometry.)

In general, the local similarity solution $w_s^{(n)}$, corresponding to the n-th term of the series in (A3), could be expressed in terms of the complex variable $z = r(cos\theta + isin\theta)$:

$$w_s^{(n)}(r,\theta) = \frac{C_n}{\pi(\lambda_n - 1)}\left[\frac{\pi}{4}r^{\lambda_n}cos(\lambda_n - 2)\theta + \mathcal{R}eal(z^{\lambda_n}logz)\right], \qquad (A11)$$

where

$$logz = log(r) + i\theta, \quad \lambda_n = 2(2n+1), \quad n = 0, 1, 2, ... \qquad (A12)$$

The second term in (A11), being the real part of an analytical function (everywhere, except at the origin), is a harmonic function. This term provides homogeneous boundary conditions on $\theta = \pm\frac{\pi}{4}$, and is responsible for the nonregularity of the similarity solution $w_s^{(n)}$ at the origin. Separating the real part of $z^{\lambda_n}logz$ in (A11) yields

$$w_s^{(n)}(r,\theta) = \frac{C_n r^{\lambda_n}}{\pi(\lambda_n - 1)}\left[\frac{\pi}{4}cos(\lambda_n - 2)\theta + log(r)cos\lambda_n\theta - \theta\sin\lambda_n\theta\right]. \qquad (A13)$$

In the Table II, we list the first terms of the series (A3) and nonregular terms of the corresponding similarity solutions.

TABLE II. Right-hand side of Poisson's equation and corresponding nonregular terms.

n	λ_n	$f^{(n)}$ - RHS of Eq. (A3)	$w_s^{(n)}$ - non-regular term
0	2	1	$r^2\ log(r)cos(2\theta)$
1	6	$r^4cos(4\theta)$	$r^6\ log(r)cos(6\theta)$
2	10	$r^8cos(8\theta)$	$r^{10}log(r)cos(10\theta)$

For $n = 1$ and $n = 2$, it can be seen that only specific high-order schemes, the sixth- and tenth-order, respectively, are affected by the right-hand side of the Poisson equation listed in the Table II.

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