A Revised Failure Criterion for Brittle Elastic Materials Under Mixed-mode Loading in 2-D

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Abstract

We revise the failure criterion by Yosibash, Priel and Leguillon for V-notched 2-D domains made of brittle elastic materials [1] under mixed-mode loading. It is based on a finite fracture mechanics (FFM) concept that requires the determination of a finite virtual crack that simultaneously satisfies the normal stress criterion and the energy release rate (ERR) criterion. The FFM ERR criterion is extended to include mode II toughness ($\mathcal{G}_{IIc}$) and reduces to the classical fracture mechanics failure criterion under constrained mode II at the limit when the V-notch solid angle approaches 2π.

The revised criterion predictions of failure load and failure initiation angle are compared to experimental observations on PMMA V-notched specimens under mixed mode loading. The predictions compared to the original criterion are slightly improved, especially as the mode mixity ratio increases.

Keywords: Mixed-mode fracture, Generalized stress intensity factors, V-notch

1. Introduction

Failure initiation criteria for brittle materials containing V-notches of variable opening angles have been extensively investigated, especially when a complex state of stress is present, because of failure initiation phenomena that may occur for example in electronic devices [2]. A comparison of several failure criteria with experimental observations was presented in [3] demonstrating their validity. We consider a 2-D domain containing a V-notch with a solid angle ω (so that the opening angle is $2\pi - \omega$) with a virtual crack of length ℓ that initiates at an angle θ_0, see Figure 1.

The various failure initiation criteria for *brittle* materials containing *sharp* V-notches of

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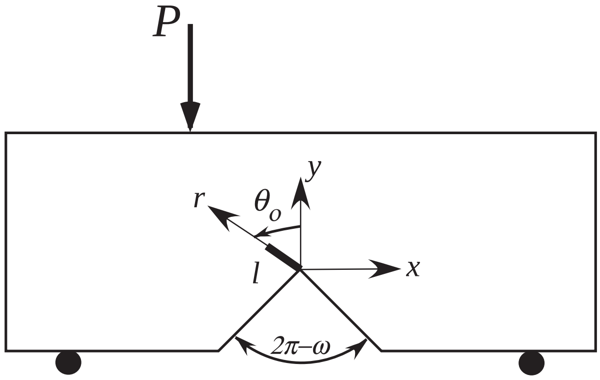

Figure 1: The V-notched specimen with a crack at its tip.

variable opening angles under a state of mixed-mode stresses are aimed at predicting the load that cause failure initiation (crack formation) in the vicinity of the V-notch tip, and some that also predict the failure initiation angle. Comparison of the predicted failure load and failure initiation angles with experimental observations have shown in most cases a good match. All these criteria are based (in addition to the Young modulus E and Poisson ratio ν) on the plane-strain mode I critical stress intensity factor K_{Ic} and the ultimate stress in tension σ_c (since brittle materials are considered the yield and ultimate stress coincide). We herein mention some of these failure criteria in a chronological order:

a) The Seweryn et al. criterion for mixed mode loading [4] postulates that failure initiates once the mean value of circumferential stress along a precalculated damage length d_0 reaches a critical value. This criterion uses a critical length d_0 which is a function of K_{Ic} and σ_c.

b) Variants of the finite fracture mechanics (FFM) criteria, that simultaneously satisfy a FFM energy release rate (ERR) constraint and a tangential stress (or stress average) constraint along a given virtual crack, are presented in [1, 5]. These also make use of a characteristic length ℓ_0 that is a function of mode I fracture toughness $\mathcal{G}_{Ic}$ (computed by K_{Ic}) and σ_c, the 1-D stress at brittle failure.

c) Mean value of the strain energy density (also denoted by averaged SED) [3, 6, 7, 8] postulates that failure occurs once the mean SED in a circular region around the V-notch tip reaches a critical value SED_{cr} which is a function of σ_c. This mean SED must be computed in a circular sector determined by another characteristic length R_{mat} determined by K_{Ic} and σ_c.

d) The sharp V-notch maximum tangential stress (SV-MTS) by [9], which has been shown to provide accurate results for PMMA at a large variety of mixed mode loading. This criterion uses K_{Ic}, and a characteristic length that is a function of σ_c. It is interesting to note that the SV-MTS criterion reduces to the well known linear elastic fracture mechanics criterion $K_I = K_{Ic}$ at the limit of a crack when pure mode I is considered, but for pure mode II, it reduces to the condition $K_{IIc} = 1.15K_{Ic}$ instead of the well accepted condition $K_{IIc} \approx 0.87K_{Ic}$.

All failure criteria mentioned in a)-c) are not applicable to pure mode II. Both the Seweryn criterion and the Yosibash, Priel & Leguillon criterion result in progressively worse predicted failure loads as the V-notch solid angle $\omega \to \pi$ because the GSIF for such large angles tends to a constant (large V-notch opening angles ($2\pi - \omega \geq 120^0$) are not often found in practice).
The failure criterion in d), although predicts well pure mode II failure, may possibly be a bit off at the limit of pure mode II since it may have a too large nonlinear zone close to the V-notch tip as suggested in [10]: " *...those by Ayatollahi et al. [17] are outside 15% small scale yielding. Here, it is clear that the small scale yielding assumption is being stretched... the analysis by Ayatollahi et al. [17] predicts failure accurately even in pure mode II conditions for* $2\alpha = 270^o$ *when our analysis suggests that the process zone is outside 15% small scale yielding*" .

To the best of our knowledge, the predictions of failure criteria associated with V-notch tips under 2D mixed mode loadings deteriorate as the mode mixity increases (mode I influence reduces progressively). We postulate that including K_{IIc} (equivalently $\mathcal{G}_{IIc}$) in the formulation of the criteria may improve their prediction capabilities in these cases when mode mixity increases. Only the variant of the FFM criteria applied to determine the failure initiation load of bonded joints in [11] considers a critical shear stress in the adhesive in addition to the critical tensile stress. It is conceivable that K_{IIc} or $\mathcal{G}_{IIc}$ should be present in such criteria, because a V-notch at the limit when the solid angle approaches 2π becomes the classical case of a crack. So, any failure criteria must coincide at the limit with the well accepted crack initiation theorem $K_{II} = K_{IIc}$ in classical linear elastic fracture mechanics when the crack is at a critical length and fracture initiates restricted to propagate along crack's initial direction. These arguments motivated us to revise the failure initiation criterion in [1] by considering $\mathcal{G}_{IIc}$.

In section 2 we present the notations and briefly recall the FFM failure initiation criterion in [1]. We then introduce the modification of the criterion by accounting for $\mathcal{G}_{IIc}$. To determine the quality of the revised criterion, we consider past mixed mode experiments conducted on V-notched specimens made of PMMA (polymer) in section 3. Comparison of predicted failure load and crack initiation angle for a large range of mode mixity ratios are presented. We summarize by conclusions in section 4.

2. The original failure criterion and its revision

2.1. Preliminaries and notations

Consider a 2-D domain under the assumption of plane-strain having a V-notch reentrant corner shown in Figure 2. The stresses in the vicinity of the sharp V-notch tip, are expressed as an asymptotic series, see e.g. [1]:

$$\boldsymbol{\sigma} \stackrel{\text{def}}{=} \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} = \sum_{i=1}^{\infty} A_i r^{\alpha_i - 1} \begin{Bmatrix} \sigma_{rr}^{(i)}(\theta) \\ \sigma_{\theta\theta}^{(i)}(\theta) \\ \sigma_{r\theta}^{(i)}(\theta) \end{Bmatrix} = \sum_{i=1}^{\infty} A_i r^{\alpha_i - 1} \boldsymbol{\sigma}^{(i)}(\theta) \tag{1}$$

where r, θ are cylindrical coordinates located in the V-notch tip , A_i is a coefficient that depends on the load called generalized stress intensity factor (GSIF), α_i the eigenvalues and $\boldsymbol{\sigma}^{(i)}(\theta)$ the eigenvectors, which depend on the geometry in the vicinity of the V-notch tip. When $r \to 0$ the first two terms dominate since $\alpha_1, \alpha_2 < 1$ and (1) is explicitly given

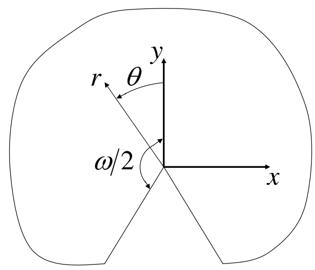

Figure 2: Coordinate system at the V-notch tip.

as:

$$\begin{aligned} \boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{r\theta} \end{Bmatrix} &= A_1 r^{\alpha_1 - 1} \begin{Bmatrix} \left[\cos(1+\alpha_1)\theta + \frac{(3-\alpha_1)}{(1-\alpha_1)} \frac{\sin[\omega(1+\alpha_1)/2]}{\sin[\omega(1-\alpha_1)/2]} \cos(1-\alpha_1)\theta\right] / \sigma_{\theta\theta}^{I}(\theta=0) \\ \left[-\cos(1+\alpha_1)\theta + \frac{(1+\alpha_1)}{(1-\alpha_1)} \frac{\sin[\omega(1+\alpha_1)/2]}{\sin[\omega(1-\alpha_1)/2]} \cos(1-\alpha_1)\theta\right] / \sigma_{\theta\theta}^{I}(\theta=0) \\ \left[-\sin(1+\alpha_1)\theta + \frac{\sin[\omega(1+\alpha_1)/2]}{\sin[\omega(1-\alpha_1)/2]} \sin(1-\alpha_1)\theta\right] / \sigma_{\theta\theta}^{I}(\theta=0) \end{Bmatrix} \\ &+ A_2 r^{\alpha_2 - 1} \begin{Bmatrix} \left[\sin(1+\alpha_2)\theta + \frac{(3-\alpha_2)}{(1+\alpha_2)} \frac{\sin[\omega(1+\alpha_2)/2]}{\sin[\omega(1-\alpha_2)/2]} \sin(1-\alpha_2)\theta\right] / \sigma_{r\theta}^{II}(\theta=0) \\ \left[-\sin(1+\alpha_2)\theta + \frac{\sin[\omega(1+\alpha_2)/2]}{\sin[\omega(1-\alpha_2)/2]} \sin(1-\alpha_2)\theta\right] / \sigma_{r\theta}^{II}(\theta=0) \\ \left[\cos(1+\alpha_2)\theta - \frac{(1-\alpha_2)}{(1+\alpha_2)} \frac{\sin[\omega(1+\alpha_2)/2]}{\sin[\omega(1-\alpha_2)/2]} \cos(1-\alpha_2)\theta\right] / \sigma_{r\theta}^{II}(\theta=0) \end{Bmatrix} \end{aligned} \tag{2}$$

with

$$\sigma^{I}_{\theta\theta}(\theta=0) \stackrel{\text{def}}{=} \frac{(1+\alpha_1)}{(1-\alpha_1)}\frac{\sin[\omega(1+\alpha_1)/2]}{\sin[\omega(1-\alpha_1)/2]} - 1 \tag{3}$$

$$\sigma^{II}_{r\theta}(\theta=0) \stackrel{\text{def}}{=} 1 - \frac{(1-\alpha_2)}{(1+\alpha_2)}\frac{\sin[\omega(1+\alpha_2)/2]}{\sin[\omega(1-\alpha_2)/2]} \tag{4}$$

chosen so to normalize the "eigen-stresses" so that for mode I

$$\sigma^{(1)}_{\theta\theta}(\theta=0) = 1$$

and for mode II:

$$\sigma^{(2)}_{r\theta}(\theta=0) = 1$$

The equation (2) represents two in plane modes: mode I which opens the V-notch faces and mode II that shears the V-notch face along $x = 0$. A cracked or V-notched body may be subject to any of these modes or a combination (named "mixed mode" state).

2.2. Original failure criterion and its flaw

The corner-stone of this criterion is the postulate that a virtual finite crack length ℓ has to be instantaneously created so to satisfy both the FFM ERR criterion and stress criterion. For a given small load, a large crack is required so as to satisfy the FFM ERR criterion, whereas only at small distance from the notch tip the tangential stress reaches σ_c (the stress criterion is satisfied). To satisfy both criteria simultaneously, the load has to be increased so that the upper bound of the virtual crack length decreases to satisfy the FFM ERR criterion, and the distance from the notch tip where tangential stress reaches σ_c increases. Only when the load is increased to a level where the lower bound of a virtual crack to satisfy the FFM ERR criterion equals the upper bound of the distance to satisfy the stress criterion, a crack of this length is crated, leading to failure, satisfying simultaneously the two criteria.

The original FFM ERR [1, eq. (35)] induces a lower limit for the virtual crack ℓ:

$$A_1^2\ell^{2\alpha_1-1}H_{11}(\omega,\theta) + A_1A_2\ell^{\alpha_1+\alpha_2-1}(H_{12}(\omega,\theta)+H_{21}(\omega,\theta)) + A_2^2\ell^{2\alpha_2-1}H_{22}(\omega,\theta) \geq \mathcal{G}_{Ic} \tag{5}$$

with $\mathcal{G}_{Ic} = \frac{K_{Ic}^2}{E/(1-\nu^2)}$ for plane strain.

The expressions H_{ij} were computed by finite element methods in [1] and tabulated for several V-notch opening angles ω for cracks that initiate at any given angles θ at the V-notch tip. The expression in (5) is a generalization of a small kinking crack from an existing crack [12] to the case of a small kinking crack from a V-notch tip. The H_{ij}s reduce to these in [12] for a kinking crack in case the V-notch becomes a crack ($\omega = 2\pi$), as demonstrated in the following subsection.

2.2.1. *The crack limit: $\omega \to 2\pi$, $\alpha_1 \to 1/2$ and $\alpha_2 \to 1/2$*

At the crack limit the FFM ERR criterion (5) reduces to:

$$A_1^2 H_{11}(2\pi,\theta) + A_1 A_2 (H_{12}(2\pi,\theta) + H_{21}(2\pi,\theta)) + A_2^2 H_{22}(2\pi,\theta) \geq \mathcal{G}_{Ic} \tag{6}$$

Using the connection $A_1 \stackrel{\text{def}}{=} K_I/\sqrt{2\pi}$, $A_2 \stackrel{\text{def}}{=} K_2/\sqrt{2\pi}$ the LHS of (6) is of an identical form as the ERR expression for a small crack kinking under mixed mode first presented in [12] (note that there the angle θ is denoted by ω and C_{12} stands for $H_{12} + H_{21}$):

$$K_1^2 C_{11}(\theta) + K_1 K_2 C_{12}(\theta) + K_2^2 C_{22}(\theta) \tag{7}$$

For example, for $E = 1$ and $\nu = 0.36$ we present in Figure 2.2.1 the computed $H_{11}(2\pi,\theta)$, $H_{12}(2\pi,\theta) + H_{21}(2\pi,\theta)$ and $H_{22}(2\pi,\theta)$ compared to these reported in [12, Table 4].

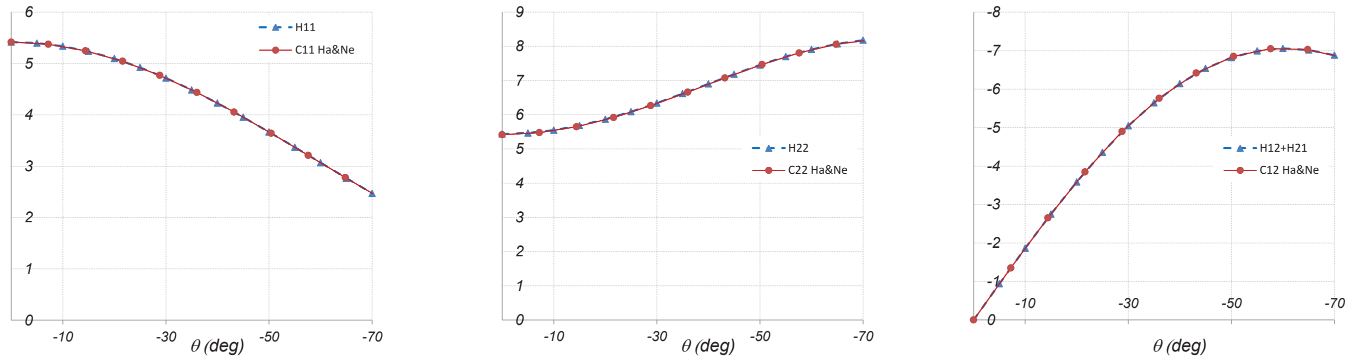

Figure 3: $H_{ij}(2\pi,\theta)$ compared to C_{ij} by Hayashi and Nemat-Nasser [12] for $E = 1$, $\nu = 0.36$.

For a pure mode I case, $A_2 \to 0$, $\theta \to 0$ so the FFM ERR (6) reduces to the classical linear elastic ERR criterion:

$$\frac{K_I^2}{2\pi} H_{11}(2\pi, 0) \geq \mathcal{G}_{Ic}. \tag{8}$$

For example, for $E = 1$ and $\nu = 0.36$ one obtains $H_{11}(2\pi, 0) = 5.41816$ so that (8) becomes

$$K_I^2 \geq \frac{2\pi}{5.41816} \frac{K_{Ic}^2 \times 1}{(1 - 0.36^2)} \approx K_{Ic}^2,$$

which is exactly the toughness fracture criterion for cracks.

However, a physical inconsistency is evident at the limit of a pure mode II, $A_1 = 0$, $A_2 \stackrel{\text{def}}{=} K_{II}/\sqrt{2\pi}$ for a self-similar crack propagation (along $\theta = 0$). In this case

$$\frac{K_{II}^2}{2\pi} H_{11}(2\pi, 0) \geq \mathcal{G}_{Ic}. \tag{9}$$

For same example as previously considered, for $E = 1$ and $\nu = 0.36$ one obtains $H_{22}(2\pi, 0) = 5.43965$ so that (9) becomes

$$K_{II}^2 \geq \frac{2\pi}{5.43965} \frac{K_{Ic}^2 \times 1}{(1 - 0.36^2)} \approx K_{Ic}^2,$$

which implies that $K_{IIc} \equiv K_{Ic}$, that is incorrect.

2.3. The revised failure criterion

None of the mixed-mode failure criteria at sharp V-notches account for critical mode II ERR $\mathcal{G}_{IIc}$. Following the desire to capture micro-mechanical effects by a macro-mechanical model, we follow concepts of phenomenological characterization of interface toughness that consider mechanisms responsible for dependence of the toughness on mode mixity via confined plasticity and constrained fracture along possible grain structure. A simplified failure criterion along these considerations is given in [13, eq. (2.4)]:

$$\frac{\mathcal{G}_I}{\mathcal{G}_{Ic}} + \frac{\mathcal{G}_{II}}{\mathcal{G}_{IIc}} = 1.$$

In [13] "*$\mathcal{G}_{IIc} = \mathcal{G}_{Ic}/\lambda$ has the interpretation as the pure mode 2 toughness.*" Since shearing a crack and opening a crack result in different physical behaviour at the instance of crack propagation, it is conceivable to account for $\mathcal{G}_{IIc}$ in the ERR. At the limit when the V-notch becomes a crack $\omega \to 2\pi$, one expects to obtain for self-similar crack initiation criterion, i.e. the the critical mode II ERR attained. To account for both $\mathcal{G}_{Ic}$ and $\mathcal{G}_{IIc}$ we consider instead of (5) the following revised FFM ERR criterion:

$$\frac{A_1^2 \ell^{2\alpha_1 - 1} H_{11}(\omega, \theta)}{\mathcal{G}_{Ic}} + \frac{A_1 A_2 \ell^{\alpha_1 + \alpha_2 - 1}(H_{12}(\omega, \theta) + H_{21}(\omega, \theta))}{\sqrt{\mathcal{G}_{Ic}\mathcal{G}_{IIc}}} + \frac{A_2^2 \ell^{2\alpha_2 - 1} H_{22}(\omega, \theta)}{\mathcal{G}_{IIc}} \geq 1 \quad (10)$$

with $\mathcal{G}_{IIc} = \frac{K_{IIc}^2}{E/(1-\nu^2)}$ for plane strain. Multiplying (10) by $\mathcal{G}_{Ic}$ one obtains

$$A_1^2 \ell^{2\alpha_1 - 1} H_{11}(\omega, \theta) + \beta A_1 A_2 \ell^{\alpha_1 + \alpha_2 - 1}(H_{12}(\omega, \theta) + H_{21}(\omega, \theta)) + \beta^2 A_2^2 \ell^{2\alpha_2 - 1} H_{22}(\omega, \theta) \geq \mathcal{G}_{Ic} \quad (11)$$

where

$$\beta \stackrel{\text{def}}{=} \sqrt{\frac{\mathcal{G}_{Ic}}{\mathcal{G}_{IIc}}} = \frac{K_{Ic}}{K_{IIc}}.$$

Remark 1. *Of course that for an ideally brittle homogeneous material with unconstrained crack initiation direction, the virtual crack is expected to initiate in a pure opening mode according to the FFM ERR criterion that coincides in this case with the maximum tangential stress criterion. This ideal case does not require therefore $\mathcal{G}_{IIc}$.*

One may notice that the revised FFM ERR criterion reduces at the limit of pure mode II and for cracks to the classical ERR criterion:

$$A_2^2 H_{22}(2\pi, 0) \geq \mathcal{G}_{Ic}/\beta^2 = \mathcal{G}_{IIc} \tag{12}$$

The upper limit for ℓ is the maximum distance from the V-notch tip at an angle θ at which the tangential stress is higher than σ_c (stress criterion):

$$\sigma_{\theta\theta}(\ell, \theta) = A_1 \ell^{\alpha_1 - 1} \sigma_{\theta\theta}^{(1)}(\theta) + A_2 \ell^{\alpha_2 - 1} \sigma_{\theta\theta}^{(2)}(\theta) \geq \sigma_c \tag{13}$$

Defining:

$$m(\ell) \stackrel{\text{def}}{=} \frac{A_2}{A_1} \ell^{\alpha_2 - \alpha_1}, \tag{14}$$

we may reformulate (11) and (13):

$$\ell \;\geq\; \left(\frac{\mathcal{G}_{Ic}}{A_1^2 \left[H_{11}(\omega, \theta) + \beta m (H_{12}(\omega, \theta) + H_{21}(\omega, \theta)) + (\beta m)^2 H_{22}(\omega, \theta) \right]} \right)^{\frac{1}{2\alpha_1 - 1}} \tag{15}$$

$$\ell \;\leq\; \left(\frac{A_1 (\sigma_{\theta\theta}^{(1)}(\theta) + m \sigma_{\theta\theta}^{(2)}(\theta))}{\sigma_c} \right)^{\frac{1}{1 - \alpha_1}} \tag{16}$$

If both FFM ERR and stress criteria have to hold at the instance of virtual crack creation then the lower and upper bound of ℓ in (15-16) have to coincide. This results in the following generalized stress intensity factors:

$$A_1^{-2} = \left(\frac{\mathcal{G}_{Ic}}{H_{11}(\omega, \theta) + \beta m (H_{12}(\omega, \theta) + H_{21}(\omega, \theta)) + (\beta m)^2 H_{22}(\omega, \theta)} \right)^{-2 + 2\alpha_1} \left(\frac{\sigma_{\theta\theta}^{(1)}(\theta) + m \sigma_{\theta\theta}^{(2)}(\theta)}{\sigma_c} \right)^{4\alpha_1 - 2} \tag{17}$$

Substituting (17) in (15), one obtains the virtual finite crack size ℓ_0 that has to be present so to satisfy (10 and 13) simultaneously:

$$\ell_0 = \frac{\mathcal{G}_{Ic}}{H_{11}(\omega, \theta) + \beta m(\ell_0)(H_{12}(\omega, \theta) + H_{21}(\omega, \theta)) + (\beta m)^2(\ell_0) H_{22}(\omega, \theta)} \left(\frac{\sigma_{\theta\theta}^{(1)}(\theta) + m(\ell_0) \sigma_{\theta\theta}^{(2)}(\theta)}{\sigma_c} \right)^2 \tag{18}$$

Note that the determination of ℓ_0 requires a solution of an implicit equation because m depends on ℓ_0.

After ℓ_0 is determined, $m(\ell_0)$ is known and can be substituted in (17) to obtain the generalized stress intensity factor (GSIF) at failure, denoted by A_{1c}:

$$A_{1c} = \left(\frac{\mathcal{G}_{Ic}}{H_{11}(\omega, \theta) + \beta m (H_{12}(\omega, \theta) + \beta H_{21}(\omega, \theta)) + (\beta m)^2 H_{22}(\omega, \theta)} \right)^{1 - \alpha_1} \left(\frac{\sigma_c}{\sigma_{\theta\theta}^{(1)}(\theta) + m \sigma_{\theta\theta}^{(2)}(\theta)} \right)^{2\alpha_1 - 1} \tag{19}$$

Remark 2. *The exponent of the first term in* (19) *has a typographical error in [1, eq. (42)] and [3, eq. (11)]. It is* $\alpha_1 - 1$ *instead of the right expression* $1 - \alpha_1$.

Remark 3. *Since* A_{1c} *depends on* θ *and* m *(which depends on* A_2/A_1 *and* ℓ_0*), for each* θ *a different value of* ℓ_0 *and* A_{1c} *is obtained. The failure initiation angle* θ_c *is the one that results in the smallest value of* A_{1c} *for a given* A_2/A_1*, and thus also* ℓ_c *is determined. Therefore one has to determine* ℓ_0 *for all values of* θ_0*, and use these* ℓ_0 *for each angle to determine the minimum value of* A_{1c}.

Remark 4. *Under mode I loading, for which* $A_2 = 0$, (19) *degenerates to the explicit expression (28) in [14].*

We now may summarize the computation algorithm to determine the critical load and failure initiation angle:

a) Construct the FE model of the domain with the V-notch (given E and ν) and load it by unit force.

b) Extract the two GSIFs A_1 and A_2.

c) Determine by experiments or database K_{Ic}, K_{IIc}, σ_c.

d) Use tabulated data given in [1] (computed for $E = 1, \nu = 0.36$) for $H_{11}(\omega, \theta)$, $H_{22}(\omega, \theta)$, $H_{12}(\omega, \theta) + H_{21}(\omega, \theta)$ and adjust to the appropriate E, ν by

$$H_{ij}^{new}(\omega, \theta) = H_{ij}(\omega, \theta) \frac{1}{1 - 0.36^2} \frac{1 - \nu^2}{E} \tag{20}$$

e) Compute ℓ_0 by (18) for different θ, then estimate $A_{1c}(\theta)$ by (19).

f) Determine the failure initiation angle to be the one that results in the minimum value of A_{1c}.

$$\min_{\theta} A_{1c}(\theta) = A_{1c}(\theta_c)$$

g) The load to failure is then computed by:

$$F_{cr} = \frac{A_{1c}(\theta_c)}{A_1}$$

3. Predicted failure load and failure initiation angle by the revised criterion compared to experimental observations

To assess whether the revised failure criterion has any advantage over the original one [1], we compare their predictions of the failure load and failure initiation angles for V-notched specimens made of PMMA. We use the experimental observations having different V-notch angles and mode mixity ratios reported in [4] (see Figure 4), as well as experiments performed by Yosibash et al. on bars with 315^o V-notches (see Figure 5). The material properties reported for the double V-notched specimens tested by Seweryn and Lukaszewicz at room temperature are [4]: $E = 3300\ MPa$, $\nu = 0.35$, $K_{Ic} = 1.202\ MPa\sqrt{m}$, $\sigma_c = 102.8\ MPa$.

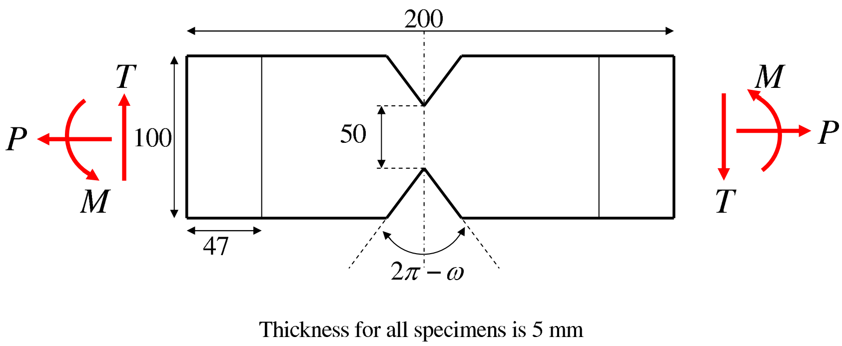

Figure 4: PMMA specimens reported in [4]: Dimensions and loading configuration (from [3]).

The 315^o V-notched bars were loaded by a non symmetric three-point-bending load to obtain a state of mixed mode at the V-notch tip, see Figure 5. These experiments were performed at $198K$ and $296K$ temperatures. At $296K$ the measured material parameters reported in [3] are $E = 3100\ MPa$, $\nu = 0.36$, $\sigma_c = 111.8\ MPa$ and $K_{Ic} = 1.03$ to 1.25 $MPa\sqrt{m}$, and at $198K$ $E = 4980\ MPa$, $\nu = 0.36$, $\sigma_c = 179.5\ MPa$ and $K_{Ic} = 2.09$ to $2.25\ MPa\sqrt{m}$.

The revised criterion requires K_{IIc} which is unavailable neither in [4] nor in [3]. To estimate its value, a literature survey on the ratio K_{IIc}/K_{Ic} for PMMA was performed, summarized in Table 1. Since the spread in β^2 is large when inspecting Table 1, we use the well accepted ratio of $\frac{1}{\beta} = \frac{K_{IIC}}{K_{IC}} = 0.866$ (resulting in $\frac{\mathcal{G}_{IIC}}{\mathcal{G}_{IC}} = 0.75 \ \rightarrow \beta^2 = 4/3$) obtained from the maximum tangential stress criterion for cracks, also known as the criterion which assures that crack propagation is along $K_{II} = 0$. The experimental data shown in Table 1 confirm that the used ratio $\frac{K_{IIC}}{K_{IC}}$ is within 30% of this theoretical value.

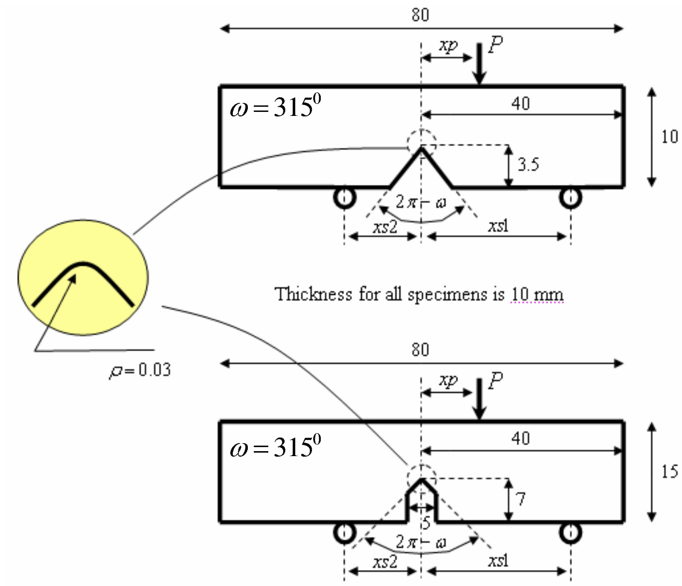

Figure 5: PMMA specimens dimensions and loading configuration (from [3]).

Table 1: Mode I and II fracture toughness for PMMA.

$K_{IC}\,[MPa\sqrt{m}]$	$K_{IIC}\,[MPa\sqrt{m}]$	K_{IIC}/K_{IC}	β^2	$E\,[GPa]$, ν	Ref.	Type of specimen/test
1.03-1.1 (Avg.1.06)	0.83-1.06 (Avg. 0.94)	Avg. 0.887	1.28	3.4 , .39	[15]	compact shear
1.26	1.4	1.11	0.81	-	[16]	diam. comp. notched Brazilian disk
Avg. 1.38	1.37-1.66 (Avg. 1.51)	Avg. 1.1	0.91	-	[17]	edge-cracked mode II fracture
1.42	1.05-1.11	0.76	1.75	3.14,3.44 , .35,.38	[18]	beam specimen with edge cracks
1.75-1.97 (Avg. 1.87)	1.77	Avg. 0.95	1.05	-	[19]	symmetric & anti-symm. 4PB
Avg. 2.1	Avg. 1.9	Avg. 0.9	1.23	2.8 ,-	[20]	single edge notch (SEN)
1.99-2.3 (Avg. 2.13)	0.9-1.36 (Avg. 1.12)	Avg. 0.52	1.92	-	[21]	semi-circular bend

Finally, using functions $H_{11}, H_{22}, H_{12}+H_{21}$ tabulated in [1], the material properties and the GSIFs, we predict the load to failure and failure initiation angles for PMMA specimens and present them as a function of the mode mixity in Figures 6 and 7. Both the revised and original criteria predictions are shown as well as the experimental observations to allow the comparison between them.

Although the revised criterion does not impose a big change on the predictions, in all cases the predicted failure initiation angle is improved. The predicted load to failure is generally better as the mode mixity increases (m increases), whereas as $m \to 0$ the revised criterion is very similar to the original one as expected. For the Seweryn&Lukaszewicz specimens with the largest solid angle $\omega = 340^o$ the experimental failure loads are by far larger compared to the predictions, which may be a result of the blunting of the crack tip because of the difficulty to manufacture V-notches having a 20^o opening with very sharp manufacturing disks.

In Table 2 we summarise the virtual crack lengths at failure ℓ_c for all experimental data. Indeed $\ell_c \ll 1$, and very minor changes are noticed in ℓ_c between the original and revised criteria, with the difference becoming larger (as expected) as the mode mixity m increases.

We also checked the sensitivity of the failure load prediction to σ_c, especially for the $\omega = 300^o$ and $\omega = 280^o$ at the highest $m = 1.151$ and $m = 0.804$ respectively. A change of $\pm 10\%$ results in a small change of $\approx \pm 3\%$ in the predicted failure load, which suggests that σ_c may be a bit too highly estimated.

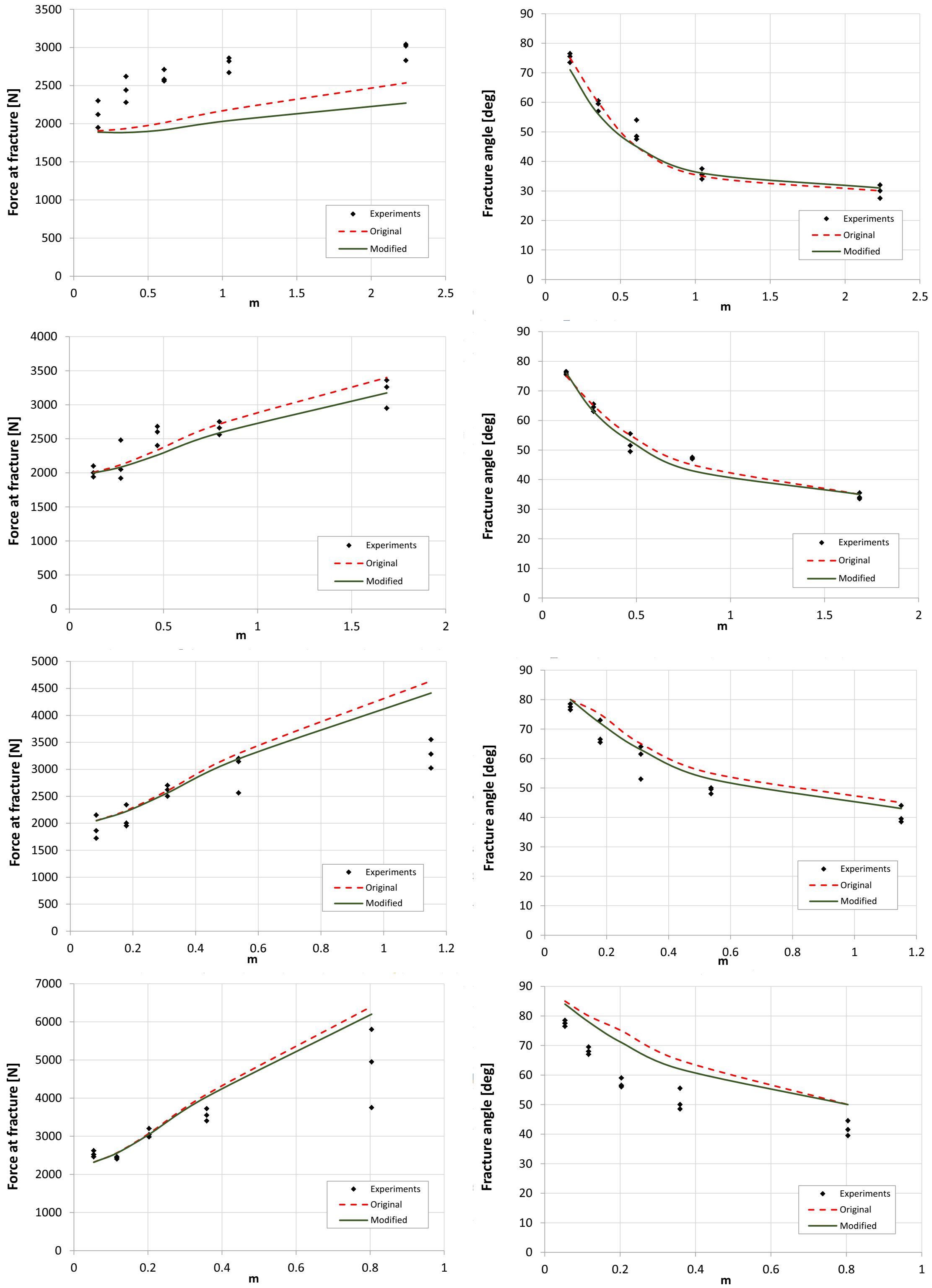

Figure 6: Predicted and experimental critical load $[N]$ and failure initiation angle for the specimens in [4] with 340^o (top), 320^o, 300^o and 280^o (bottom) V-notch angles.

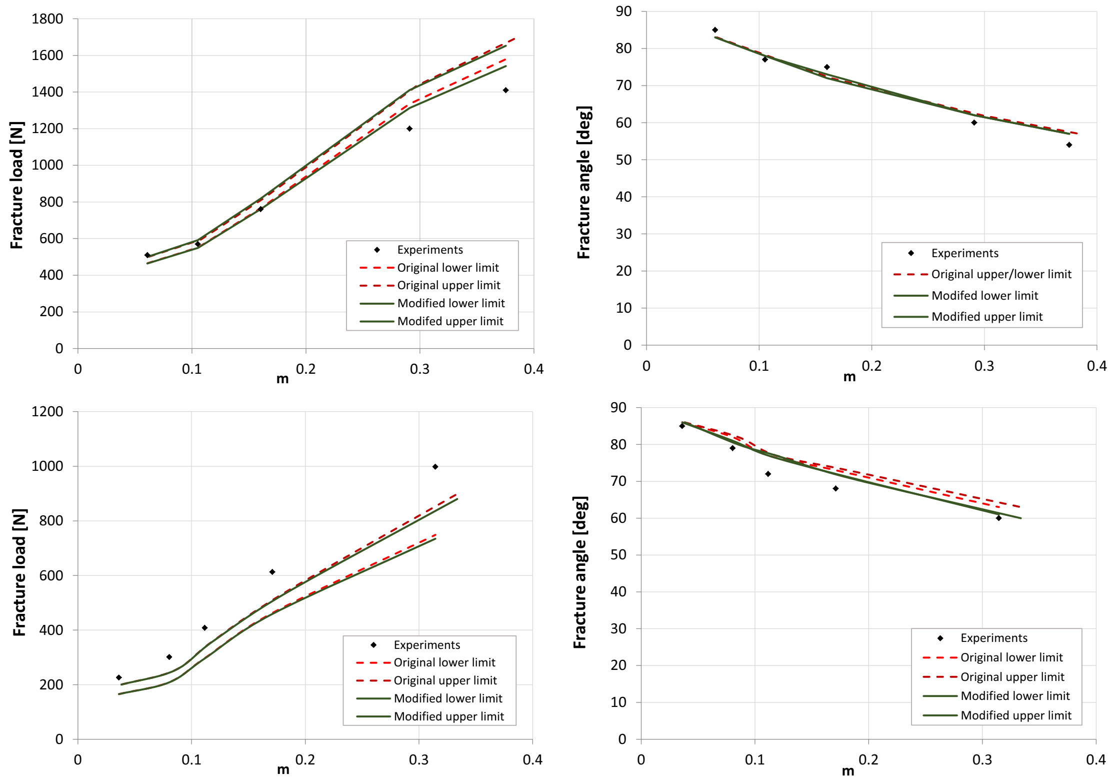

Figure 7: Predicted and experimental critical load $[N]$ and failure initiation angle for PMMA specimens [3] with 315^o at 198K (upper row) and 296K (lower row) V-notch angles.

Table 2: Virtual crack length ℓ_c (mm) computed by the original and revised failure criteria.

	m (mode mixity)	ℓ_c original	ℓ_c revised
	0.036	0.0138	0.0138
	0.080	0.0138	0.0137
Bar $\omega = 315^o$, $296K$: $K_{Ic} = 1.03\,MPa\sqrt{m}$	0.112	0.0137	0.0136
	0.171	0.0135	0.0133
	0.314	0.0129	0.0123
	0.038	0.0204	0.0204
	0.085	0.0203	0.0201
Bar $\omega = 315^o$, $296K$: $K_{Ic} = 1.25\,MPa\sqrt{m}$	0.119	0.0201	0.0200
	0.182	0.0199	0.0194
	0.334	0.0190	0.0180
	0.061	0.0221	0.022
	0.105	0.0219	0.0217
Bar $\omega = 315^o$, $198K$: $K_{Ic} = 2.09\,MPa\sqrt{m}$	0.160	0.0217	0.0213
	0.291	0.0207	0.0199
	0.375	0.0203	0.0190
	0.062	0.0256	0.0255
	0.108	0.0254	0.0252
Bar $\omega = 315^o$, $198K$: $K_{Ic} = 2.25\,MPa\sqrt{m}$	0.164	0.0251	0.0246
	0.298	0.0240	0.0230
	0.384	0.0235	0.0220
	0.164	0.0218	0.0203
	0.352	0.0205	0.0168
Double V-notch $\omega = 340^o$	0.608	0.0190	0.0133
	1.045	0.0169	0.0102
	2.233	0.0148	0.0076
	0.127	0.0218	0.0216
	0.272	0.0205	0.0198
Double V-notch $\omega = 320^o$	0.467	0.0190	0.0176
	0.797	0.0169	0.0150
	1.687	0.0148	0.0121
	0.083	0.0228	0.0227
	0.179	0.0229	0.0224
Double V-notch $\omega = 300^o$	0.310	0.0228	0.0219
	0.537	0.0226	0.0208
	1.151	0.0220	0.0191
	0.054	0.0243	0.0242
	0.116	0.0246	0.0245
Double V-notch $\omega = 280^o$	0.203	0.0253	0.0249
	0.359	0.0269	0.0259
	0.804	0.0304	0.0275

4. Summary and Conclusions

The FFM failure criterion for V-notch tips in 2-D domains under mixed mode loading in [1] has been revised to include the $\mathcal{G}_{IIc}$ material parameter. It is to the best of our knowledge the first time the mode II fracture toughness is considered for V-notched specimens. The predictions were compared to experimental observations on PMMA specimens from past publications. Because the $\mathcal{G}_{IIc}$ was not measured in these past experiments and a large scatter is reported for PMMA in other experiments, we estimated the value of $\beta^2 = \mathcal{G}_{Ic}/\mathcal{G}_{IIc} \approx 4/3$.

The predicted failure initiation angles using the revised criterion are always closer to the experimental results compared to the original criterion, although the changes are small. The predicted failure loads by the revised criterion were not always closer to the experimental observations, however the prediction improves for larger V-notched opening angles and as the mode mixity increases. Although no big changes have been observed in the predictions, the overall quality of the predicted failures was improved. Thus, it is conceivable that any failure criterion should include the mode II fracture toughness, although more data is involved in the failure criterion.

Revising our former criterion we noticed some misprints and inaccuracies in [1] and [3] which were corrected here.

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References

[1] Z. Yosibash, E. Priel, and D. Leguillon. A failure criterion for brittle elastic materials under mixed-mode loading. *Int. Jour. Fracture*, 141(1):291–312, 2006.

[2] Z. Yosibash, O. Adan, R. Shneck, and H. Atlas. Thermo-mechanical failure criterion at the micron scale in electronic devices. *Int. Jour. Fracture*, 122:47–64, 2003.

[3] E. Priel, A. Bussiba, I. Gilad, and Z. Yosibash. Mixed mode failure criteria for brittle elastic V-notched structures. *Int. Jour. Fracture*, 144:247–265, 2007.

[4] A. Seweryn and A. Lukaszewicz. Verification of brittle fracture criteria for elements with V-shaped notches. *Engrg. Frac. Mech.*, 69:1487–1510, 2002.

[5] A. Sapora, P. Cornetti, and A. Carpinteri. A finite fracture mechanics approach to V-notched elements subjected to mixed-mode loading. *Engrg. Frac. Mech.*, 97:216 – 226, 2013.

[6] P. Lazzarin, F. Berto, M. Elices, and J. Gomez. Brittle failures from U- and V-notches in mode I and mixed, I + II, mode: a synthesis based on the strain energy density averaged on finite-size volumes. *Fat. Frac. Engrg. Mat. Struct.*, 32(8):671–684, 2009.

[7] F.J. Gomez, M. Elices, F. Berto, and P. Lazzarin. Fracture of V-notched specimens under mixed mode (I+II) loading in brittle materials. *Int. Jour. Fracture*, 159:121–135, 2009.

[8] F. Berto and P. Lazzarin. Recent developments in brittle and quasi-brittle failure assessment of engineering materials by means of local approaches. *Materials Science and Engineering: R: Reports*, 75:1 – 48, 2014.

[9] M.R. Ayatollahi, A.R. Torabi, and P. Azizi. Experimental and theoretical assessment of brittle fracture in engineering components containing a sharp V-notch. *Experimental Mechanics*, 51:919–932, 2011.

[10] R.C. Flicek, D.A. Hills, and D. Dini. Refinements in the characterisation of mode-mixity and small scale yielding at sharp notch roots. *Engrg. Frac. Mech.*, 126:73 – 86, 2014.

[11] P. Weisgraeber and W. Becker. Finite fracture mechanics model for mixed mode fracture in adhesive joints. *Int. Jour. Solids and Structures*, 50(1415):2383 – 2394, 2013.

[12] K. Hayashi and S. Nemat-Nasser. Energy release rate and crack kinking under combined loading. *Trans. ASME, Jour. Appl. Mech.*, 48(3):520–524, 1981.

[13] J.W. Hutchinson and Z. Suo. Mixed mode cracking in layered materials. In J.W. Hutchinson and T.Y. Wu, editors, *Advances in Applied Mechanics*, volume 29, pages 63–191. Academic Press Inc., 1992.

[14] D Leguillon. Strength or toughness? A criterion for crack onset at a notch. *Eur. Jour. Mech. A - Solids*, 21(1):61–72, 2002.

[15] L. Banks and M. Arcan. A compact mode II fracture specimen. *American Soc. Test. Mater.*, 7:347–363, 1986.

[16] H. Awaji, T. Kato, S. Honda, and T. Nishikawa. Criterion for combined mode I-II of brittle fracture. *Jour. Ceramic Soc. Japan*, 107:918–924, 1999.

[17] L. Banks-Sills and M. Arcan. Mode II fracture testing with application to PMMA. In *Application of fracture mechanics to materials and structures*. Martinus Nijhoff, 1984.

[18] Y. Ueda, K. Ikeda, T. Yao, and M. Aoki. Characteristics of brittle fracture under general combined modes including those under bi-axial tensile loads. *Engrg. Frac. Mech.*, 18:1131–1158, 1983.

[19] T.M. Maccagno and J.F. Knott. The fracture behavior of PMMA in mixed modes I and II. *Engrg. Frac. Mech.*, 34:65–86, 1989.

[20] J.C.W. Davenport and D.J. Smith. A study of superimposed fracture modes I,II and III on PMMA. *Fat. Frac. Engrg. Mat. Struct.*, 16:1125–1133, 1993.

[21] M.R. Ayatollahi, M.R.M. Aliha, and M.M. Hassani. Mixed mode brittle fracture in PMMA - an experimental study using SCB specimens. *Materials Science and Engineering A*, 417:348–356, 2006.